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Initial-value problem for Maxwell and linearized Einstein fields

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Abstract. We prove that if $F_{ab}=-F_{ba}$ is a bivector field on Minkowskian space–time satisfying the wave equation, $\Box F_{ab}=0$, and the derivatives of F_{ab} satisfy certain smoothness and asymptotic conditions, then one can construct from F_{ab} a bivector which satisfies Maxwell's vacuum field equations which, from another point of view, are equivalent to polarization conditions. The relationship between this result and the initial-value problem is solved by two different techniques. We also provide the extension of these results to the linearized Einstein field equations *in vacuo*.

1. Introduction

While a substantial amount is known concerning the initial-value problem for the scalar wave equation in Minkowskian space–time (see Pounder and Synge 1955, Courant and Hilbert 1962, John 1971 and references therein) the special problems encountered when dealing with the analogous problem for the vacuum Maxwell or linearized Einstein field equations have received little attention (cf Courant and Hilbert 1962, p 647). In studying this problem we found that the essential ingredient was the fact that the field (the Maxwell field§ F_{ab} or linearized Einstein field R_{abcd}) in each case must satisfy the wave equation||

$$\Box F_{ab}=0 \qquad \text{and} \qquad \Box R_{abcd}=0. \tag{1.1}$$

This is, of course, a consequence of the vacuum Maxwell equations

$$F_{ab,b}=0, \qquad F_{[ab,c]}=0, \tag{1.2}$$

and the vacuum linearized Einstein equations

$$R_{bc}=R_{abca}=0, \qquad R_{ab[cd,e]}=0. \tag{1.3}$$

§ Latin indices run 1, 2, 3, 4. We choose coordinates X_a for which (X_1, X_2, X_3) are rectangular Cartesians and $X_4=\mathrm{i}t$ (units are chosen for which $c=G=1$) so that the metric tensor of Minkowskian space–time is the Kronecker delta δ_{ab}. The Einstein summation convention is used and square brackets denote antisymmetrization.

|| At this stage we merely suppose F_{ab} and R_{abcd} to have the *algebraic* symmetries

$$F_{ab}=-F_{ba}, \qquad R_{abcd}=R_{cdab}=-R_{bacd}, \qquad R_{a[bcd]}=0,$$

thus we include the Bianchi identities (the second of (1.3)) in the set of field equations, to preserve the symmetry between (1.2) and (1.3).

However it raises the question, to what extent the content of (1.2) and (1.3) is embodied in (1.1). We provide an answer to this question in § 2 by proving that if R_{bc}, $F_{ab,b}$ and their first derivatives satisfy certain asymptotic and smoothness conditions then (1.1) *imply* that there exist tensor fields (which can be constructed out of F_{ab} and R_{abcd}) which satisfy (1.2) and (1.3). In § 3 we solve the initial-value problem for the Maxwell equations following a method used by Pounder and Synge (1955) for the scalar wave equation, which utilizes a complex wavefunction (an alternative technique is described in appendix 2). We indicate how the procedure is also used in solving the initial-value problem for the linearized Einstein equations. The central role being played by (1.1) and the necessity for the theorem proved in § 2 will then be obvious. In § 4 we describe how (1.2) and (1.3) may be interpreted as polarization conditions and thus the content of Maxwell's or of the linearized Einstein equations, above and beyond the wave equation (1.1), is clarified. The paper ends with a discussion of our results in § 5.

2. Two useful theorems

The difficulty in passing from (1.1) to (1.2) and (1.3) without putting unreasonable constraints on F_{ab} and R_{abcd} is illustrated by the following two theorems. In them we investigate the effect of placing asymptotic restrictions on $F_{ab,b}$ and R_{bc} which have been used by Synge (1965, p 412), in the bivector case, to derive a variational principle from which one can obtain Maxwell's equations.

Theorem 1. Let $F_{ab}=-F_{ba}$ be a bivector field on Minkowskian space–time with continuous derivatives existing at least to order three. Let $F_{ab,b}$ and its first derivatives vanish at least as fast as $r^{-1-\alpha}$ $(\alpha>0)$ on every null-cone at every event in Minkowskian space–time. If, in addition, F_{ab} satisfies the wave equation, $\Box F_{ab}=0$, then we can construct out of F_{ab} a bivector K_{ab} which satisfies Maxwell's vacuum field equations (1.2).

Proof. Under the stated smoothness and asymptotic conditions (cf Synge 1965, p 412) Synge has proved that F_{ab} may be decomposed into

$$F_{ab}=H_{ab}+K_{ab}, \tag{2.1}$$

where

$$H_{ab,b}=0, \qquad K_{[ab,c]}=0. \tag{2.2}$$

Hence

$$F_{ab,b}=K_{ab,b} \qquad F_{[ab,c]}=H_{[ab,c]}. \tag{2.3}$$

Now let F_{ab} satisfy the wave equation so that

$$\Box H_{ab}=-\Box K_{ab}. \tag{2.4}$$

It follows from (2.2) that

$$\Box K_{ab,b}=0. \tag{2.5}$$

Now invoking a well known theorem based on Kirchhoff's formula (cf Fock 1964, p 365) we conclude from (2.5) and the asymptotic conditions on $F_{ab,b}$

$$K_{ab,b}=0, \tag{2.6}$$

and thus, by (2.2) and (2.6), K_{ab} satisfies the Maxwell equations (1.2).

The method of constructing K_{ab} out of F_{ab} is given explicitly in Synge's proof of the decomposition (2.1) (Synge 1965, p 412).

We may extend this result to cover the linearized Einstein field equations (1.3) as follows.

Theorem 2. Let R_{abcd} be a tensor field on Minkowskian space–time having the algebraic symmetries given in the second footnote on p 899, having continuous derivatives up to third order and having R_{bc}, and its first derivatives vanish at least as fast as $r^{-1-\alpha}$ $(\alpha>0)$ on every null-cone at every event in Minkowskian space–time and $(R_{bc}-\frac{1}{2}\delta_{bc}R_{dd})_{,c}=0$. If, further, R_{abcd} satisfies the wave equation, $\Box R_{abcd}=0$, then there exists a K_{abcd} which can be constructed from a knowledge of R_{abcd} and satisfies the linearized Einstein field equation (1.3).

Proof. Under the stated asymptotic and smoothness conditions we prove, in appendix 1, a similar result to that of Synge quoted in theorem 1, namely, that R_{abcd} may be decomposed into

$$R_{abcd}=H_{abcd}+K_{abcd}, \tag{2.7}$$

where

$$H_{bc}\equiv H_{abca}=0, \qquad K_{ab[cd,e]}=0. \tag{2.8}$$

Hence

$$R_{bc}=K_{bc}\equiv K_{abca}, \qquad R_{ab[cd,e]}=H_{ab[cd,e]}. \tag{2.9}$$

Now let R_{abcd} satisfy the wave equation so that

$$\Box H_{abcd}=-\Box K_{abcd}, \tag{2.10}$$

from which, using (2.8), we have

$$\Box K_{bc}=0. \tag{2.11}$$

Again, we invoke a well known theorem based on Kirchhoff's formula (Fock 1964, p 365) to conclude from (2.11), on the basis of the asymptotic conditions assumed on R_{bc}, that

$$K_{bc}=0. \tag{2.12}$$

Hence, by (2.8) and (2.12), K_{abcd} satisfies the linearized Einstein field equations (1.3) and K_{abcd} is constructed out of R_{abcd} in the manner of appendix 1.

3. The initial-value problem

We shall here describe the solution of the initial-value problem for Maxwell's equations (1.2) and outline a similar approach for the case of (1.3). We follow the point of view of

Pounder and Synge (1955) by first introducing the complex wavefunction

$$w = S^{-1}, \qquad S = (X_a - \alpha_a - i\beta_a)(X_a - \alpha_a - i\beta_a), \tag{3.1}$$

where α_a, β_a are fixed four-vectors and β_a is time-like ($\beta_a\beta_a = -\beta^2$) and, in fact, to simplify matters we choose $\beta_a = (0, i\beta)$. The wavefunction (3.1) vanishes at large distances from the coordinate origin but is singularity free (it is discussed in Synge 1965, p 360). Consider the tensor

$$Q_{abc} = wF_{ab,c} - w_{,c}F_{ab}. \tag{3.2}$$

This has the property that

$$Q_{abc,c} = w\Box F_{ab}, \tag{3.3}$$

which vanishes (almost everywhere) if and only if F_{ab} satisfies the wave equation. This provides us with a useful conservation law with which to solve the initial-value problem for a bivector field F_{ab} satisfying the wave equation (an alternative approach is described in appendix 2). In doing this, since Maxwell's equations (1.2) imply the first of (1.1) we are, in fact, solving the initial-value problem for a wider class of bivector fields than Maxwellian fields. Theorem 1 of the previous section illustrates how one might try to recover Maxwell's equations without putting unreasonable conditions on F_{ab}.

Since the treatment which follows of the initial-value problem parallels the argument of Pounder and Synge (1955) we merely sketch the argument here. The reader may refer to Pounder and Synge for details.

Applying Green's theorem to $Q_{abc,c} = 0$ in the four-volume indicated in figure 1 we obtain

$$\int_{t=\alpha} \left(w\frac{\partial F_{ab}}{\partial t} - \frac{\partial w}{\partial t}F_{ab}\right) d\sigma = \int_{\mathrm{BCDE}} \left(w\frac{\partial F_{ab}}{\partial n} - \frac{\partial w}{\partial n}F_{ab}\right) d\sigma. \tag{3.4}$$

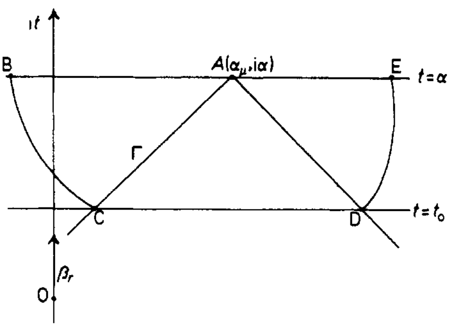

Figure 1. The four-volume to which we apply Green's theorem is that bounded by the space-like three-flat $t = \alpha$ between B and E and the three-surface $BCDE$. Γ is the past null-cone with vertex at the event $A(\alpha_a)$.

Multiply this equation by i, take the real part and then the limit $\beta \to 0$. We then obtain

$$\lim_{\beta\to 0} \mathrm{Re}\left(i\int_{t=\alpha} \frac{\partial w}{\partial t}F_{ab}\, d\sigma\right) = \lim_{\beta\to 0} \mathrm{Re}\left[i\int_{\mathrm{BCDE}} \left(\frac{\partial w}{\partial n}F_{ab} - w\frac{\partial F_{ab}}{\partial n}\right) d\sigma\right]. \tag{3.5}$$

With w given by (3.1) we can easily prove that the left-hand side of this equation is simply F_{ab} evaluated at $A(\alpha_a)$. We can also see that the only contribution to the right-hand side of (3.5) comes from the intersection of the past null-cone Γ (see figure 1) and the three-surface BCDE (this is a manifestation of Huygen's principle). Thus (3.5) gives

$$F_{ab}(\alpha_\mu, \mathrm{i}\alpha) = \lim_{\beta\to 0} \mathrm{Re}\left[\mathrm{i}\int_{t=t_0}\left(\frac{\partial w}{\partial t}F_{ab} - w\frac{\partial F_{ab}}{\partial t}\right)\mathrm{d}\sigma\right]. \tag{3.6}$$

Evaluating the integral in the manner of Pounder and Synge we arrive at the Kirchhoff formula

$$F_{ab}(\alpha_\mu, \mathrm{i}\alpha) = \left[\frac{\partial}{\partial r}(rP_{ab}) + rQ_{ab}\right]_{\substack{t=t_0\\ r=\tau_0}}, \tag{3.7}$$

where $r^2 = (X_\mu - \alpha_\mu)(X_\mu - \alpha_\mu)$, $(\mu = 1, 2, 3)$, $\tau = \alpha - t$ with $\tau_0 = \alpha - t_0$ and

$$P_{ab} = (4\pi)^{-1}\int F_{ab}\,\mathrm{d}\omega, \qquad Q_{ab} = (4\pi)^{-1}\int \frac{\partial F_{ab}}{\partial t}\,\mathrm{d}\omega, \tag{3.8}$$

where $\mathrm{d}\omega$ is the element of solid angle on a two-sphere $r = \text{constant}$. Equation (3.7) expresses the solution to the initial-value problem for the first of (1.1) in that it expresses F_{ab} at A in terms of F_{ab} and $\partial F_{ab}/\partial t$ given on the initial space-like hypersurface $t = t_0$.

Theorem 1 of the previous section establishes conditions under which a bivector K_{ab}, constructed from F_{ab}, will satisfy Maxwell's equations. Being in possession of equation (3.7) we may now ask to pass from it to a solution of the initial-value problem for K_{ab}. This can be achieved as follows: from (2.1) we have

$$K_{ab}(\alpha_\mu, \mathrm{i}\alpha) = F_{ab}(\alpha_\mu, \mathrm{i}\alpha) - H_{ab}(\alpha_\mu, \mathrm{i}\alpha), \tag{3.9}$$

and, by (2.4), H_{ab} satisfies

$$\Box H_{ab} = h_{ab}, \qquad h_{ab} = -\Box K_{ab}. \tag{3.10}$$

By Synge's (1965) construction of K_{ab} we have in addition

$$h_{ab} = F_{bc,ca} - F_{ac,cb}. \tag{3.11}$$

By the first of (2.3) in conjunction with (2.6) this expression vanishes under the conditions of theorem 1. Hence F_{ab} and K_{ab} differ by a solution of the homogeneous wave equation, with vanishing divergence. If we now further assume that

$$H_{ab} = F_{ab} - K_{ab} = \mathrm{O}(r^{-1-\alpha}), \qquad \alpha > 0, \tag{3.12}$$

on every null-cone at every event in Minkowskian space–time and that first derivatives of H_{ab} behave similarly then it follows that $H_{ab} = 0$ and by (3.9), (3.7) represents a solution to the initial-value problem for the Maxwellian field K_{ab}.

An exactly similar treatment may be used for the tensor field R_{abcd}. We define the tensor

$$T_{abcde} = wR_{abcd,e} - w_{,e}R_{abcd}, \tag{3.13}$$

from which we obtain

$$T_{abcde,e} = w\Box R_{abcd}, \tag{3.14}$$

and this vanishes (almost everywhere) if and only if R_{abcd} satisfies the wave equation.

Solving the initial-value problem as above for $R_{abcd}(\alpha_a)$ we obtain

$$R_{abcd}(\alpha_\mu, \mathrm{i}\alpha) = \left[\frac{\partial}{\partial r}(rP_{abcd}) + rQ_{abcd}\right]_{\substack{t=t_0\\ r=\tau_0}}, \tag{3.15}$$

where

$$P_{abcd} = (4\pi)^{-1}\int R_{abcd}\,\mathrm{d}\omega, \qquad Q_{abcd} = (4\pi)^{-1}\int \frac{\partial R_{abcd}}{\partial t}\,\mathrm{d}\omega. \tag{3.16}$$

As in the Maxwellian case we have here solved the initial-value problem for a wider class of fields than vacuum linearized Einstein fields while theorem 2 of the previous section illustrates how one might recover a linearized Einstein field K_{abcd}. The solution of the initial-value problem for K_{abcd} is obtained from (3.11) and the results of theorem 2 in exactly similar fashion as in the Maxwellian case detailed above for K_{ab}.

4. Polarization conditions

In this section we analyse the physical content of the Maxwell and vacuum linearized Einstein equations (1.2) and (1.3), over and above the equations of wave propagation (1.1) of those respective fields.

Firstly we will cast (1.2) and (1.3) into a form that manifests their formal equivalence and then we will trace them back to the same source—that they are both nothing other than polarization conditions on the wave-propagating fields (1.1).

Following the procedure in Tchrakian (1975) we express the two fields F_{ab} and R_{abcd} as

$$F_{ab} = e^{\mu}_{ab}(E_\mu + \mathrm{i}H_\mu), \tag{4.1}$$

$$R_{abcd} = e^{\mu\nu}_{abcd}(E_{\mu\nu} + \mathrm{i}H_{\mu\nu}), \tag{4.2}$$

where the indices μ, ν run over 1, 2, 3 in three-dimensional Euclidean space. The rotational tensors $E_{\mu\nu}$ and $H_{\mu\nu}$ are symmetric and traceless and the symmetries and explicit expressions for the coefficients e^{μ}_{ab} and $e^{\mu\nu}_{abcd}$ are given in Tchrakian (1975, equation (3)).

Substituting F_{ab} and R_{abcd} into (1.2) and (1.3), we get the following 'Maxwell equations':

$$\begin{aligned} &\nabla\cdot\boldsymbol{E} = 0, \qquad &&\nabla\times\boldsymbol{E} = -\partial\boldsymbol{H}/\partial t,\\ &\nabla\cdot\boldsymbol{H} = 0, \qquad &&\nabla\times\boldsymbol{H} = \partial\boldsymbol{E}/\partial t, \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} &\nabla\cdot\boldsymbol{E}_\nu = 0, \qquad &&(\nabla\times\boldsymbol{E}_\mu)_\nu = -\partial H_{\mu\nu}/\partial t,\\ &\nabla\cdot\boldsymbol{H}_\nu = 0, \qquad &&(\nabla\times\boldsymbol{H}_\mu)_\nu = \partial E_{\mu\nu}/\partial t, \end{aligned} \tag{4.4}$$

where the notation in (4.4) is that of Tchrakian (1975). Equations (4.3) and (4.4) manifest the formal equivalence of (1.2) and (1.3).

At this point we mention that (4.4) are exactly equations (11′) of Tchrakian (1975), which were said to be derived from the contracted form of the Bianchi identities, i.e.

$$R_{ab[cd,a]} = 0, \tag{4.5}$$

while in our case they are derived from (1.3) itself. There is no discrepancy here, as (4.4) and (4.5) can readily be verified, using (4.2), to be equivalent in the case of vacuum linearized Einstein fields.

It is clear that vacuum 'Maxwell equations' similar to (4.3) and (4.4) exist also for tensor fields of arbitrary rank.

The origin of both (4.3) and (4.4) are the polarization constraints on the wave (1.1). To arrive at this conclusion we have to take recourse to the particle–wave duality, whereby a wave propagating at the speed of light is interpreted as a massless particle. Now it is well known that massless particle wavefunctions must satisfy what are called 'unitarity constraints' (cf Zwanziger 1964, Weinberg 1965, Niederer and O'Raifeartaigh 1974) whose function is to express that *helicity* is the physical observable, and of course the classical analogue of helicity is the polarization of the wave propagating at the speed of light (Jackson 1975, pp 273, 333).

There are many diverse but equivalent expressions for the unitarity conditions (Jackson 1975) and the most suitable in our notation is

$$W_a\psi = \lambda p_a\psi, \tag{4.6}$$

where $W_a = \frac{1}{2}\epsilon_{abcd}p_bM_{cd}$ is the Pauli–Lubanski vector, p_a is the four-momentum of the massless particle and M_{cd} is the generator of Lorentz transformations in the representation carried by the wavefunction or field ψ (cf Weinberg 1975), e.g. ψ can be F_{ab} or R_{abcd}, and λ is the helicity.

The condition (4.6), with the four-momentum replaced by space–time derivatives, can be shown to lead (Niederer 1975, Dunne 1976) to the Bianchi identities of (1.2) and (1.3) or to the 'Maxwell equations' (4.3) and (4.4).

We can thus see that a wave field propagating with the speed of light will satisfy certain polarization constraints if it satisfies equations (1.2) and (1.3).

5. Summary

We have seen in § 3 how the problem of passing from (1.1) to (1.2) and (1.3) arises in solving the initial-value problem for (1.2) and (1.3). Using a result due to Synge we have studied this passage in § 2. We considered placing the same asymptotic conditions on $F_{ab,b}$ and R_{bc} as Synge finds necessary to place on $F_{ab,b}$ in order to obtain a variational principle which will yield Maxwell's equations. We then find that (1.1) allow us, not to pass directly to (1.2) and (1.3) but to construct tensor fields K_{ab} and K_{abcd} out of F_{ab} and R_{abcd} respectively which do satisfy (1.2) and (1.3). In § 3 we have sketched a solution to the initial-value problems for (1.1) and have described how one passes from these solutions to the solutions of the initial-value problems for K_{ab} and K_{abcd}. The physical content of (1.2) and (1.3) over and above (1.1) is studied in § 4 and we conclude that (1.2) and (1.3) are equivalent to polarization conditions on the wave fields (1.1).

Finally, we may point out that the method of Lichnerowicz (cf Adler *et al* 1965) for studying the initial-value problem for the Einstein or Einstein–Maxwell equations may also be applied to the equations we have been discussing in this paper. In his method one quotes F_{ab} and/or g_{ab} and $\partial g_{ab}/\partial t$ on the initial hypersurface and then one uses the field equations to obtain higher-order time derivatives on the initial hypersurface of these field quantities in terms of the given data. The field at a time to the future of the initial hypersurface is obtained by Taylor expansion of the field quantities about the initial hypersurface. Some of the field equations impose constraints on the allowable initial data. The method we have described in this paper has the advantages that the field is determined: (*a*) by an explicit formula in terms of the initial data; and (*b*) at any time to the future of the initial hypersurface (whereas the Lichnerowicz approach may be restricted by the region of convergence of the Taylor expansion). The disadvantage in our approach is brought about by not solving the initial-value problem directly for (1.2) and (1.3) but by proceeding via the wave equations (1.1). Then the passage back to (1.2) and (1.3) requires asymptotic conditions to be placed on the fields which do not appear in the Lichnerowicz approach.

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Appendix 1

Let R_{abcd} satisfy the algebraic symmetries in the second footnote on p 899 and the asymptotic and smoothness conditions of theorem 2. Let $\gamma_{ab}=\gamma_{ba}$ be such that $\gamma^*_{ab,b}=0$ where $\gamma^*_{ab}=\gamma_{ab}-\frac{1}{2}\delta_{ab}\gamma_{cc}$. Define

$$K_{abcd}=\tfrac{1}{2}(\gamma_{ac,bd}+\gamma_{bd,ac}-\gamma_{ad,bc}-\gamma_{bc,ad}), \tag{A.1}$$

then

$$K_{ab[cd,e]}=0. \tag{A.2}$$

Now define

$$H_{abcd}=R_{abcd}-K_{abcd}, \tag{A.3}$$

then, in particular,

$$H^*_{bc}=R^*_{bc}-K^*_{bc}, \tag{A.4}$$

and

$$H^*_{bc,c}=R^*_{bc,c}=0 \tag{A.5}$$

since $K^*_{bc,c}=0$. We may also write (A.4) as

$$H^*_{bc}=R^*_{bc}+\tfrac{1}{2}\Box\gamma^*_{bc}, \tag{A.6}$$

and thus if $H_{bc}=0$ we have the following equations for γ_{ab} given R_{abcd}:

$$\Box\gamma^*_{bc}=-2R^*_{bc}, \qquad \gamma^*_{bc,c}=0. \tag{A.7}$$

Under the conditions on R_{abcd} of theorem 2 these equations have a solution γ_{ab} and hence, by (A.1), K_{abcd} exists and consequently, by (A.3), H_{abcd} exists.

Appendix 2

We describe here an alternative method (cf John 1971) for solving the initial-value problem for Maxwell's equations.

Let

$$P_{ab}(r, \mathrm{i}t) = (4\pi)^{-1} \int_{\xi_\mu \xi_\mu = 1} F_{ab}(\alpha_\mu + r\xi_\mu, \mathrm{i}t)\, \mathrm{d}\omega. \tag{A.8}$$

Assuming F_{ab} to be continuous it is clear from (A.8) that

$$\lim_{r \to 0} P_{ab}(r, \mathrm{i}\alpha) = F_{ab}(\alpha_\mu, \mathrm{i}\alpha). \tag{A.9}$$

Now following a procedure clearly outlined in John (1971) one can prove that

$$\frac{\partial^2}{\partial r^2}(rP_{ab}) = \frac{\partial^2}{\partial t^2}(rP_{ab}), \tag{A.10}$$

provided

$$\Box F_{ab} = 0. \tag{A.11}$$

We may write out the general solution to the two-dimensional wave equation (A.10) as

$$P_{ab}(r, \mathrm{i}t) = \frac{(r+\tau)P_{ab}(r+\tau, t_0) + (r-\tau)P_{ab}(r-\tau, t_0)}{2r} + (2r)^{-1} \int_{r-\tau}^{r+\tau} \eta \left[\frac{\partial P_{ab}(\eta, t)}{\partial t}\right]_{t=t_0} \mathrm{d}\eta, \tag{A.12}$$

and taking the limit of this indicated in (A.9) we reproduce the expression (3.7) for $F_{ab}(\alpha_\mu, \mathrm{i}\alpha)$. Again we see clearly here that only the wave equation (A.11) is used, and this is a consequence of, but not equivalent to, the Maxwell equations (1.2).

Similarly we may take

$$P_{abcd}(r, \mathrm{i}t) = (4\pi)^{-1} \int_{\xi_\mu \xi_\mu = 1} R_{abcd}(\alpha_\mu + r\xi_\mu, \mathrm{i}t)\, \mathrm{d}\omega, \tag{A.13}$$

and we have

$$\lim_{r \to 0} P_{abcd}(r, \mathrm{i}\alpha) = R_{abcd}(\alpha_\mu, \mathrm{i}\alpha), \tag{A.14}$$

and we can prove that rP_{abcd} must satisfy the two-dimensional wave equation (A.10) if R_{abcd} satisfies

$$\Box R_{abcd} = 0. \tag{A.15}$$

Writing out the general solution to the two-dimensional wave equation and taking the limit indicated in (A.14) we can reproduce the formula (3.11).

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